Journal of Statistical Physics, Vol. 26, No. 3, 1981

On the Absence of Spontaneous Breakdown of Continuous Symmetry for Equilibrium States in Two Dimensions

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Received March 24, 1981

Using the Bogoliubov inequality, we extend previously known results concerning the absence of continuous symmetry breakdown for equilibrium states of certain quantum and classical lattice, and continuum systems in two space dimensions.

KEY WORDS: Bogoliubov inequality; symmetry breaking.

1. INTRODUCTION

A well-known theorem of Mermin and Wagner[1] based on the Bogoliubov inequality shows that the translation-invariant equilibrium states of the two-dimensional quantum Heisenberg model with Hamiltonian $H = -\sum J(i-j)\sigma_i \cdot \sigma_j$ does not possess spontaneous magnetization if $\sum |i|^2 \cdot |J(i)| < \infty$. The same method was applied by Mermin[2] to obtain the same result for classical spins. Garrison, Morrison, and Wong[3] used the Bogoliubov inequality to show the absence of spontaneous breakdown of symmetry (rather than only the absence of spontaneous magnetization) for translation-invariant (or mildly inhomogenous) equilibrium states for these and continuum classical and quantum models. For classical lattice systems with finite-range interactions, Dobrushin and Shlosman[4] showed that every equilibrium state is invariant under the continuous symmetry. We extend the results of Ref. 3 by dropping assumptions concerning

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0022-4715/81/1100-0505$03.00/0

translation invariance of the state, and we extend the results of Ref. 4 by dropping the assumption of finite range and including quantum systems. Our results apply also to local quantum fields in two space dimensions at nonzero temperature. Similar methods apply to one-dimensional systems where, however, for lattice systems one has under similar hypotheses the stronger result of uniqueness of the equilibrium state. (See for example Refs. 5 and 6.)

On the other hand, it is known that for certain classical and quantum spin systems with couplings falling off sufficiently slowly, there does exist spontaneous symmetry breakdown.[7,8] Furthermore, phase transitions for systems with continuous symmetry and finite-range interactions can occur *without* a spontaneous symmetry breakdown.[9]

Our method is based on the Bogoliubov inequality, which applies to equilibrium states of classical and quantum lattice and continuum systems. (See for example Ref. 10.) For our purposes we will take equilibrium state to mean a state satisfying the Bogoliubov inequality, and we will show that for such states in either two or one space dimensions with sufficient falloff of the interaction there is no spontaneous breakdown of a continuous symmetry. We wish to emphasize the particular method of estimation and so we will not dwell on technical matters not related to the estimate.

2. GENERAL FORMULATION

The system under consideration will be defined on the continuum $\mathbb{R}^2$ or the lattice $\mathbb{Z}^2$. To each bounded region Λ is associated a set of observables $\mathfrak{A}_\Lambda$. The set of local observables is $\mathfrak{A} = \cup_\Lambda \mathfrak{A}_\Lambda$. There is a continuous one-parameter group σ_s of (global) symmetry transformations of $\mathfrak{A}$ such that for each finite Λ, $\sigma_s \mathfrak{A}_\Lambda = \mathfrak{A}_\Lambda$. The state ω is invariant under the symmetry group if for all local observables $\omega(\sigma_s A) = \omega(A)$. By the group property this is equivalent to

$$\left.\frac{d}{ds}\right|_{s=0} \omega(\sigma_s A) = 0$$

Given any bounded region Λ_0 and any $\epsilon > 0$ we will construct a one-parameter group $\tilde{\sigma}_s$ of transformations of $\mathfrak{A}$ such that

(i) $\sigma_s|_{\mathfrak{A}_{\Lambda_0}} = \tilde{\sigma}_s|_{\mathfrak{A}_{\Lambda_0}}$

(ii) $\tilde{\sigma}_s|_{\mathfrak{A}_\Lambda} =$ identity, if Λ is contained in the complement of Λ_1, where Λ_1 is some bounded set containing Λ_0

(iii) $\displaystyle \left| \left.\frac{d}{ds}\right|_{s=0} \omega(\tilde{\sigma}_s A) \right| \leqslant \epsilon\omega\left(\frac{A^+A + AA^+}{2} \right) \qquad \text{for all } A \in \mathfrak{A}_{\Lambda_0}$

Since ϵ may be chosen arbitrarily small it follows from (i) and (iii) that $d/ds|_{s=0}\omega(\sigma_s A) = 0$ for all $A \in \mathfrak{A}_{\Lambda_0}$. Since this holds for all Λ_0 it follows that ω is invariant under the symmetry group.

The estimate (iii) will follow from Bogoliubov's inequality, which one may write in the form[10]

$$\left| \frac{d}{ds}\bigg|_{s=0} \omega(\tilde{\sigma}_s A) \right|^2 \leqslant \beta\omega(K)\omega\left(\frac{A^+A + AA^+}{2} \right)$$

where

$$K = \frac{d}{ds}\bigg|_{s=0} \frac{d}{dt}\bigg|_{t=0} \tilde{\sigma}_s \tilde{\sigma}_t H$$

The operator K is well defined by (ii) and appropriate properties of the Hamiltonian H.

2.1. Lattice Case

For each $j \in \mathbb{Z}^2$ let $\sigma_s(j)$ be the action of the symmetry group at the site j, so that $\sigma_s = \otimes_{j \in \mathbb{Z}^2} \sigma_s(j)$. Given $c : \mathbb{Z}^2 \to \mathbb{R}$, let $\tilde{\sigma}_s = \otimes_{j \in \mathbb{Z}^2} \sigma_{c(j)s}(j)$. Then

$$\omega(K) = \sum_{i \in \mathbb{Z}^2} \sum_{j \in \mathbb{Z}^2} c(i)c(j)J(i,j)$$

where

$$J(i,j) = \frac{d}{ds}\bigg|_{s=0} \frac{d}{dt}\bigg|_{t=0} \omega(\sigma_s(i)\sigma_t(j)H)$$

Clearly $J(i,j)$ is a measure of the interaction between the spin at site i and the spin at site j. If $c(j) = 1$ for $j \in \Lambda_0$ then $\tilde{\sigma}_s|_{\mathfrak{A}_{\Lambda_0}} = \sigma_s|_{\mathfrak{A}_{\Lambda_0}}$ and if $c(j) = 0$ for j in the complement of Λ_1, then $\tilde{\sigma}_s|_{\mathfrak{A}_\Lambda} =$ identity for $\Lambda \subset \Lambda_1^c$. We will choose c so as to make $\omega(K)$ small. The properties of $J(i,j)$ which we require are the following.

Properties of $J(i,j)$:

(i) $J(i,j) = J(j,i)$
(ii) $\sum_j J(i,j) = 0$
(iii) There exists a function $f : \mathbb{Z}^2 \to \mathbb{R}$ such that $|J(i,j)| \leqslant f(i-j)$ and $\sum_j |j|^2 f(j) < \infty$.

Property (i) follows from the commutativity of $\sigma_s(i)$ and $\sigma_t(j)$. Property (ii) follows from the invariance of the Hamiltonian under the symmetries σ_s since

$$\sum_i J(i,j) = \frac{d}{ds}\bigg|_{s=0} \frac{d}{dt}\bigg|_{t=0} \omega(\sigma_t(j)\sigma_s H)$$

and $\sigma_s H = H$. Property (iii) is an assumption about the structure of the Hamiltonian. As an example, consider a quantum spin system. Let J_j be the self-adjoint operator which generates $\sigma_s(j)$, so that for $A \in \mathfrak{A}_{\{j\}}$, $\sigma_s(j)A = e^{isJ_j}Ae^{-isJ_j}$. To each finite $X \subset \mathbb{Z}^2$ is associated the $|X|$-body interaction $h(X) \in \mathfrak{A}_X$, such that for each $i \in \mathbb{Z}^2$, $\sum_{X \ni i}\|h(X)\| < \infty$. The invariance of the Hamiltonian under the symmetry transformation is expressed by $\sigma_s h(X) = h(X)$ for all X. Then

$$\sum_i J(i,j) = \sum_{X \ni j} \omega\left(\left[J_j, \frac{d}{ds}\Big|_{s=0} \sigma_s h(X)\right]\right) = 0$$

and

$$|J(i,j)| \leqslant 4\|J_i\|\,\|J_j\| \sum_{X \ni i,j} \|h(X)\| = 4\|J_0\|^2 \sum_{X \ni i,j} \|h(X)\|$$

Note that this estimate for $J(i,j)$ depends only on the interaction and not on the state ω. [If we deal with unbounded operators $h(X)$ then such an estimate on $J(i,j)$ can still be obtained using suitable hypotheses on the state ω.] If we suppose the Hamiltonian (but not the state) is translation invariant then

$$\sum_{X \ni i,j} \|h(X)\| = \sum_{X \ni 0, j-i} \|h(X)\|$$

and

$$\sum_j |j|^p \sum_{X \ni 0,j} \|h(X)\| \leqslant \sum_{X \ni 0} |X|\,|DX|^p \|h(X)\|$$

where $|X|$ is the number of lattice sites in X and DX is the diameter of $X = \sup_{i,j \in X}|i-j|$. These estimates yield (iii) if we suppose $\sum_{X \ni 0}|X|\,|DX|^2\|h(X)\| < \infty$.

A similar discussion applies to the classical case, where now we need an estimate for the second mixed derivative of $h(X)$. In Section 3 we will prove the following theorem.

Theorem A. Let $J(i,j)$ satisfy the properties (i), (ii), (iii) above. Given Λ_0 and $\epsilon > 0$ there exists $c : \mathbb{Z}^2 \to \mathbb{R}$ such that

(a) $0 \leqslant c(j) \leqslant 1 \quad \forall j \in \mathbb{Z}^2$

(b) $c(j) = 1 \quad$ for $j \in \Lambda_0$

(c) $c(j) = 0 \quad$ for $j \in \Lambda_1^c$ for some bounded $\Lambda_1 \supset \Lambda_0$

(d) $|\sum_{i,j} c(i)c(j)J(i,j)| < \epsilon$

2.2. Continuum Case

In the quantum case there will be a self-adjoint current $J(\mathbf{x})$ such that if $A \in \mathfrak{A}_\Lambda$, $d/ds|_{s=0}\sigma_s A = i[J(c), A]$, where $J(c) = \int d^2\mathbf{x}\, c(\mathbf{x})J(\mathbf{x})$ and $c(\mathbf{x}) = 1$ for $\operatorname{dist}(\mathbf{x}, \Lambda) < \delta$. The one-parameter group $\tilde{\sigma}_s$ of the previous discussion is given by $\tilde{\sigma}_s B = e^{isJ(c)} B e^{-isJ(c)}$. Clearly $\tilde{\sigma}_s B = B$ if $B \in \mathfrak{A}_{\Lambda(c)}$, since $e^{isJ(c)} \in \mathfrak{A}_{\Lambda(c)}$, where $\Lambda(c) = \{\mathbf{x} : \operatorname{dist}(x, \operatorname{supp} c) < \delta\}$ and $[\mathfrak{A}_{\Lambda_1}, \mathfrak{A}_{\Lambda_2}] = 0$ if $\Lambda_1 \cap \Lambda_2 = \emptyset$. We may write

$$\omega(K) = -i\omega\big(\big[J(c), \dot{J}(c)\big]\big) = -i\int d^2\mathbf{x}\, d^2y\, c(\mathbf{x}) c(\mathbf{y}) J(\mathbf{x}, \mathbf{y})$$

where

$$J(\mathbf{x}, \mathbf{y}) = -i\omega\big(\big[J(\mathbf{x}), \dot{J}(\mathbf{y})\big]\big) \quad \text{and} \quad \dot{J}(\mathbf{y}) = \frac{d}{dt}\bigg|_{t=0} e^{itH} J(\mathbf{y}) e^{-itH}$$

In the case of local relativistic quantum fields at nonzero temperature there will be a conserved current $(J^0(t, \mathbf{x}), \mathbf{J}(t, \mathbf{x}))$ which generates the continuous symmetry. Define $J(\mathbf{x}) = \int dt\, d^2\mathbf{y}\, \eta(t) h(\mathbf{x} - \mathbf{y}) J^0(t, \mathbf{y})$, where η is a smooth function with support in $\{|t| < \delta/2\}$, $\int dt\, \eta(t) = 1$, and $h(\mathbf{x})$ is a smooth function with support in $\{|\mathbf{x}| < \delta/2\}$, $\int d^2\mathbf{x}\, h(\mathbf{x}) = 1$. Then $J(\mathbf{x}, \mathbf{y})$ is a smooth function which vanishes if $|\mathbf{x} - \mathbf{y}| > 2\delta$ by locality (the speed of light $= 1$). In the case of nonrelativistic quantum fields the properties of $J(\mathbf{x}, \mathbf{y})$ depend on the potential coupling different points. The classical case may be formulated in a similar way with Poisson brackets replacing commutators.

The properties of $J(\mathbf{x}, \mathbf{y})$ which we shall require are

(i) $J(\mathbf{x}, \mathbf{y})$ is measurable
(ii) $J(\mathbf{x}, \mathbf{y}) = J(\mathbf{y}, \mathbf{x})$
(iii) $\int d^2\mathbf{y}\, J(\mathbf{x}, \mathbf{y}) = 0$
(iv) There exists a function $f: \mathbb{R}^2 \to \mathbb{R}$ such that $|J(\mathbf{x}, \mathbf{y})| \leqslant f(\mathbf{x} - \mathbf{y})$ and $\int d^2\mathbf{x}\, (1 + |\mathbf{x}|^2) f(\mathbf{x}) < \infty$.

In Section 3 we will prove the following theorem.

Theorem B. Let $J(\mathbf{x}, \mathbf{y})$ satisfy the properties (i), (ii), (iii), (iv) above. Given $\epsilon > 0$ there exists $c: \mathbb{R}^2 \to \mathbb{R}$ such that

(a) $0 \leqslant c(\mathbf{x}) \leqslant 1 \quad \forall \mathbf{x} \in \mathbb{R}^2$
(b) $c(\mathbf{x}) = 1 \quad$ for $|\mathbf{x}| \leqslant L$
(c) $c(\mathbf{x}) = 0 \quad$ for $|\mathbf{x}| \geqslant L'$ for some $L' > L$
(d) c is smooth (infinitely differentiable)
(e) $\left|\int d^2\mathbf{x}\, d^2\mathbf{y}\, c(\mathbf{x}) c(\mathbf{y}) J(\mathbf{x}, \mathbf{y})\right| < \epsilon$

3. THE ESTIMATE

3.1. Lattice Case

Theorem A follows from Theorem B; that is, the construction of the required c in the lattice case follows from the construction in the continuum case. There is a natural embedding of $\mathbb{Z}^2$ in $\mathbb{R}^2$ so that $j \in \mathbb{Z}^2$ is the center of a unique open unit square $\Delta_j \in \mathbb{R}^2$ and $\cup \overline{\Delta}_j = \mathbb{R}^2$. Given $J(i,j)$ with the required properties, we define a piecewise constant $J(x,y)$ by

$$J(x,y) = \begin{cases} J(i,j) & \text{if } x \in \Delta_i, \quad y \in \Delta_j \\ 0 & \text{if either } x \text{ or } y \text{ is on the boundary} \\ & \text{of a unit square} \end{cases}$$

It is not difficult to see that $J(x,y)$ has the required properties for Theorem B, and so we may construct $c(\mathbf{x})$. Then define $c(j) = \int_{\Delta_j} d^2\mathbf{x}\, c(\mathbf{x})$. Clearly $c(j)$ has all the required properties. Indeed, $c(j) = 1$ for $\mathbf{x} \in \Lambda_0$ if L is chosen large enough. Also

$$\left|\sum_{i,j} c(i)c(j)J(i,j)\right| = \left|\int d^2\mathbf{x}\, d^2\mathbf{y}\, c(\mathbf{x})c(\mathbf{y})J(\mathbf{x},\mathbf{y})\right| < \epsilon$$

3.2. Translation-Invariant Continuum Case

We consider first the case $J(\mathbf{x},\mathbf{y}) = J(\mathbf{x}-\mathbf{y})$. Then

$$\tilde{J}(\mathbf{p}) = \int d^2\mathbf{x} \exp(-i\mathbf{p}\cdot\mathbf{x})J(\mathbf{x}) = \int d^2\mathbf{x}\,(\cos\mathbf{p}\cdot\mathbf{x} - 1)J(\mathbf{x})$$

Thus

$$|\tilde{J}(\mathbf{p})| \leqslant \tfrac{1}{2}|\mathbf{p}|^2 \int d^2\mathbf{x}\, |\mathbf{x}|^2 |J(\mathbf{x})| = \alpha |\mathbf{p}|^2$$

and

$$\left|\int d^2\mathbf{x}\, d^2\mathbf{y}\, c(\mathbf{x})c(\mathbf{y})J(\mathbf{x}-\mathbf{y})\right| = \left|\frac{1}{(2\pi)^2}\int d^2\mathbf{p}\, |\tilde{c}(\mathbf{p})|^2 \tilde{J}(\mathbf{p})\right|$$

$$\leqslant \alpha \frac{1}{(2\pi)^2}\int d^2\mathbf{p}\, |\mathbf{p}|^2 |\tilde{c}(\mathbf{p})|^2 = \alpha \int d^2\mathbf{x}\, |\nabla c|^2$$

We must then construct $c(\mathbf{x})$ so that

(a) $0 \leqslant c(\mathbf{x}) \leqslant 1$
(b) $c(\mathbf{x}) = 1$ for $|\mathbf{x}| \leqslant L$
(c) $c(\mathbf{x}) = 0$ for $|\mathbf{x}| \geqslant L'$
(d) c is smooth
(e) $\int d^2\mathbf{x}\, |\nabla c|^2 < \epsilon$

We will construct c as a function of $r = |\mathbf{x}|$ only.

Let

$$a(\mathbf{x}) = \begin{cases} 1 & \text{if } |\mathbf{x}| \leqslant R \\ \left(\dfrac{|\mathbf{x}|}{R}\right)^{-\epsilon} & \text{if } |\mathbf{x}| > R \end{cases}$$

where $\epsilon > 0$. Then $\int d^2\mathbf{x}\,|\nabla a|^2 = \pi\epsilon$. Furthermore, defining $b(\mathbf{x}) = \max(2a(\mathbf{x}) - 1, 0)$ it is clear that

$$b(\mathbf{x}) = \begin{cases} 1 & \text{for } |\mathbf{x}| \leqslant R \\ 0 & \text{for } |\mathbf{x}| \geqslant R' \end{cases} \qquad \text{for some } R' > R$$

and $\int d^2x\,\|\nabla b\|^2 \leqslant 4\int d^2\mathbf{x}\,\|\nabla a\|^2 = 4\pi\epsilon$.

Finally let $c(\mathbf{x}) = \int d^2\mathbf{y}\,h(\mathbf{x}-\mathbf{y})b(\mathbf{y})$, where $0 \leqslant h(\mathbf{x}) \leqslant 1$, $\int d^2\mathbf{x}\,h(\mathbf{x}) = 1$, $h(\mathbf{x}) = 0$ if $|\mathbf{x}| > \delta$, $h(\mathbf{x})$ is infinitely differentiable and depends only on $|\mathbf{x}|$. Clearly c has properties (a)–(d) and

$$\int d^2\mathbf{x}\,|\nabla c|^2 = \frac{1}{(2\pi)^2}\int d^2\mathbf{p}\,|\mathbf{p}|^2|\tilde{c}(\mathbf{p})|^2 = \frac{1}{(2\pi)^2}\int d^2\mathbf{p}\,|\tilde{h}(\mathbf{p})|^2|\mathbf{p}|^2|\tilde{b}(\mathbf{p})|^2$$
$$\leqslant \frac{1}{(2\pi)^2}\int d^2\mathbf{p}\,|\mathbf{p}|^2|\tilde{b}(\mathbf{p})|^2 = \int d^2\mathbf{x}\,|\nabla b|^2$$

3.3. General Continuum Case

We must estimate $\int d^2\mathbf{x}\int d^2\mathbf{y}\,c(\mathbf{x})c(\mathbf{y})J(\mathbf{x},\mathbf{y})$. Using $J(\mathbf{x},\mathbf{y}) = J(\mathbf{y},\mathbf{x})$ and $\int d^2\mathbf{x}\,J(\mathbf{x},\mathbf{y}) = 0$ we have

$$\int d^2\mathbf{x}\int d^2\mathbf{y}\,c(\mathbf{x})c(\mathbf{y})J(\mathbf{x},\mathbf{y}) = -\frac{1}{2}\int d^2\mathbf{x}\int d^2\mathbf{y}\,[c(\mathbf{x}) - c(\mathbf{y})]^2 J(\mathbf{x},\mathbf{y})$$

Therefore, since $|J(\mathbf{x},\mathbf{y})| \leqslant f(\mathbf{x}-\mathbf{y})$,

$$\begin{aligned}&\left|\int d^2\mathbf{x}\int d^2\mathbf{y}\,c(\mathbf{x})c(\mathbf{y})J(\mathbf{x},\mathbf{y})\right| \\ &\quad\leqslant \frac{1}{2}\int d^2\mathbf{x}\int d^2\mathbf{y}\,[c(\mathbf{x}) - c(\mathbf{y})]^2 f(\mathbf{x}-\mathbf{y}) \\ &\quad= \int d^2\mathbf{x}\,c(\mathbf{x})^2\int d^2\mathbf{y}\,f(y) - \int d^2\mathbf{x}\int d^2\mathbf{y}\,c(\mathbf{x})c(\mathbf{y})f(\mathbf{x}-\mathbf{y}) \\ &\quad= \frac{1}{(2\pi)^2}\int d^2\mathbf{p}\,|\tilde{c}(\mathbf{p})|^2[\tilde{f}(\mathbf{o}) - \tilde{f}(\mathbf{p})]\end{aligned}$$

Now

$$\tilde{f}(\mathbf{o}) - \tilde{f}(\mathbf{p}) = \int d^2\mathbf{x}\,(1 - \cos\mathbf{p}\cdot\mathbf{x})f(\mathbf{x}) = 2\int d^2\mathbf{x}\,\sin^2(\tfrac{1}{2}\mathbf{p}\cdot\mathbf{x})f(\mathbf{x})$$

Thus $|\tilde{f}(\mathbf{o}) - \tilde{f}(\mathbf{p})| \leqslant \frac{1}{2}|\mathbf{p}|^2\int d^2\mathbf{x}\,|\mathbf{x}|^2 f(\mathbf{x}) = \gamma|\mathbf{p}|^2$. We may thus write

$$\left|\int d^2\mathbf{x}\int d^2\mathbf{y}\,c(\mathbf{x})c(\mathbf{y})J(\mathbf{x},\mathbf{y})\right| \leqslant \gamma\int d^2\mathbf{x}\,|\nabla c(\mathbf{x})|^2$$

We may now use the result of Section 3.2.

4. THE ONE-DIMENSIONAL CASE

We briefly mention here the application of our method in the one-dimensional case. The method of Section 3.3 applies, except that we now use the estimate

$$\int dx \sin^2(px) f(x) \leqslant |p| \int dx\,|x|\,f(x)$$

if $\int dx\,|x|\,f(x) < \infty$ or

$$\int dx \sin^2(px) f(x) \leqslant |p| \int dy \sin^2 y \, \frac{f((1/p)y)}{p^2} = |p| \int dy \, \frac{\sin^2 y}{p^2 + y^2}$$

$$\leqslant |p| \int dy \, \frac{\sin^2 y}{y^2} \qquad \text{if } f(x) = \frac{1}{1 + x^2}$$

We thus obtain $|\int dx \int dy\, c(x) c(y) J(x, y)| \leqslant \gamma \int dp\,|p|\,|\tilde{c}(p)|^2$.

As in Section 3.2, one may find a suitable $c(x)$ such that $\int dp\,|p|\,|\tilde{c}(p)|^2$ is as small as desired. Thus if $\int dx\,|x|\,f(x) < \infty$ or $f(x) = \alpha/(1 + x^2)$ there is no spontaneous continuous symmetry breakdown.

ACKNOWLEDGMENT

We thank B. Simon for a reformulation leading to a simplification of our original proof in Section 3.3.

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